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PROBLEM SIZE IN LINEAR PROGRAMMING

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ABSTRACT

This is a progress report covering recent attempts to estimate the size of linear programming problems. Such estimates could be very useful as a guide to practical computation and as an aid in choosing among alternate methods of solution. However, the results at present are far from complete, and many unsolved problems remain.

1. INTRODUCTION

As linear programming has developed into a powerful and popular tool for the solution of practical problems, a sizeable gap has appeared between the theory and the technique of the subject. Much more space in the literature has been devoted to routine applications or minor modifications of the standard methods than to critical examination or comparison of these metnods. In particular, there is a striking contrast between the observed practical efficiency of the simplex algorithm and a complete lack of theorems to explain this efficiency. In the words of Dantzig (3): "The simplex method amounts to moving along the edges of a convex polytope, from one vertex to a neighboring vertex which gives the greatest change in the value of the linear objective function, and continuing this process until a vertex is reached at which the function attains its optimal value. For polytopes defined by m equations in n non-negative variables, the number of such moves is remarkably low, often between m and n in practice. Intuitively, wandering along the edges of a polytope in this way would appear to be extremely inefficient, yet experimental evidence from thousands of cases is to the contrary. Why?"

Dantzig's question suggests some basic problems concerning the facial structure of convex polytopes. These problems are of considerable geometric interest and they may also be of practical

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importance, for while the simplex method and its relatives usually work well in practice Saaty (26) has commented that "...a number of large linear programming problems have been left unsolved because, after many hours of machine operation, it was not known how much longer the process would continue." Further, there are related areas such as integer programming in which the practical situation is clearly unsatisfactory, and in these a thorough geometric understanding would probably lead to significant practical improvements.

Hoping to narrow the gap mentioned above, I have recently begun a critical study of the geometrical background of linear programming, with emphasis on estimating the size of a linear programming problem --- that is, on geometrical aspects of the problem related to the computational difficulty which may be encountered in solving it. None of the important questions has been fully answered but some progress has been made. This report summarizes the results obtained thus far, in the hope of drawing attention to the questions and leading toward their ultimate solution.

2. PROGRAMMING AND POLYHEDRA

A linear programming problem is that of maximizing or minimizing a linear function φ, the <u>objective function</u>, subject to a finite number of linear constraints. The constraints define the so-called <u>feasible region</u> of the problem. They may be linear equalities, corresponding to hyperplanes in the finite-dimensional real vector space on which φ is defined, or linear inequalities corresponding to closed halfspaces, or a mixture of equalities and inequalities. In any case,

the feasible region is the intersection of a finite number of closed halfspaces. Such a set is called here a polyhedron, and a bounded polyhedron is a polytope. Dimensions are indicated by prefixes and the 0-faces, 1-faces and (d-1)-faces of a d-polyhedron are called respectively its vertices, edges and facets. Two vertices are said to be adjacent provided they are joined by an edge. Attention is restricted to pointed polyhedra, that is, to those having at least one vertex. A pointed d-polyhedron is said to be simple provided each of its vertices is on exactly d edges.

A polyhedron is said to be of class (d,n) provided it is a pointed d-polyhedron with precisely n facets. It may be difficult to determine the exact class of a feasible region from the defining constraints, but the form of the constraints does impose some immediate limitations on the class. A region defined by n linear inequality constraints in d real variables is a polyhedron of dimension at most d and has at most n facets. A region defined by m linear equality constraints in n nonnegative variables is a polyhedron of dimension at most n, is of dimension at most $n - m$ if the linear functions appearing in the equalities are linearly independent, and has at most n facets. Thus for the study of polyhedra in connection with linear programming it seems reasonable to group the polyhedra according to class and to study the behavior, with respect to feasible regions of a given class, of the notions and procedures of linear programming. And since the feasible region is sometimes known to be bounded it seems reasonable to devote some special attention to the polytopes of a given class.

Now consider the problem of maximizing a linear function on a pointed polyhedron P. A solution of this problem is defined as a vertex v of P such that either $\varphi(v) = \sup \varphi P$ or v is the endpoint of an unbounded edge E of P with $\sup \varphi E = +\infty$. A solution must exist, and indeed from any vertex x_0 of P there is a sequence $(x_0, x_1, \cdots, x_\ell)$ of vertices of P such that x_ℓ is a solution and the following conditions are satisfied:

(1) x_i is adjacent to x_{i-1} $(1 \leq i \leq \ell)$;

(2) $\varphi(x_0) < \varphi(x_1) < \cdots < \varphi(x_\ell)$.

A sequence $(x_0, x_1, \cdots, x_\ell)$ of vertices of P is called a path provided it satisfies (1) and a φ-path provided (1) and (2) are both satisfied; the number ℓ is the length of the path. The path $(x_0, x_1, \cdots, x_\ell)$ is said to be simple provided $x_i \neq x_j$ for $i \neq j$.

The most common procedures for the solution of linear programming problems are based upon various rules for the construction of φ-paths. Having found a vertex of the feasible region, one applies the rule to produce a φ-path leading from that vertex to a solution. For a given rule the required computation time is roughly proportional to the length of the resulting φ-path and thus there is considerable interest in relating this length to the class of the feasible region. The expected value of the length is of prime interest, but at present it seems impossible to give a useful mathematical definition of this notion. Computational experience has been reported by Dantzig (4, p. 156), Kuhn & Quandt (22),

Quandt & Kuhn (24, 25), and Wolfe & Cutler (27). The <u>maximum</u> length is also of interest, and some results have been obtained concerning it. They are reported in the final section below.

While a φ-path on a polytope P generally encounters only a small fraction of the vertices of P, for each class of polytopes there are simple examples in which all vertices are encounted (Klee [15, 16]). Thus it is of interest to determine the maximum number of vertices for polyhedra or polytopes of a given class. The same question arises in connection with the problem of solving completely a system of linear inequalities. And in contemplating the use of search processes more general than those involving φ-paths, one is led to ask for the maximum length of simple paths on polyhedra or polytopes of a given class. Results in these directions are summarized in the next section below.

The distance $\delta_P(x,y)$ between two vertices x and y of a polyhedron P is defined as the length of shortest path joining x and y, and the <u>diameter</u> $\delta(P)$ of P is the maximum of the distances between its vertices. Let $\Delta(d,n)$ and $\Delta_b(d,n)$ denote the maximum of $\delta(P)$ as P ranges respectively over all polyhedra of class (d,n) and all polytopes of class (d,n). For any polyhedron P it is possible to specify an objective function φ and an initial vertex so that at least $\delta(P)$ iterations are required to solve the corresponding linear programming problem, regardless of the edge-following algorithm employed. Thus $\Delta(d,n)$ represents, in a sense, the number of iterations required to solve the "worst" linear

program of n inequalities in d variables using the "best" edge-following algorithm. Results on Δ and Δ_b are reported in the second section below.

3. NUMBER OF VERTICES

Let $\mu(d,n)$ denote the maximum number of vertices of polyhedra of class (d,n). It is known (Klee [14]) that this maximum is attained only for certain simple polytopes of class (d,n), and (Gale [8]) that

$$(3) \qquad \mu(d,n) \geq \binom{n-\left\lfloor\frac{d+1}{2}\right\rfloor}{n-d} + \binom{n-\left\lfloor\frac{d+2}{2}\right\rfloor}{n-d}.$$

Indeed, the right-hand member of (3) is the number of vertices of the simple polytopes of class (d,n) which are polar to cyclic d-polytopes. Several authors have conjectured that equality always holds in (3) (Motzkin [23], Jacobs & Schell [12], Gale [8]). The conjecture has been proved by Gale [9] for $n \leq d + 3$ and by Klee [14] for $n \geq (d/2)^2 - 1$. These two results show that it is valid for $d \leq 6$, a result obtained also by Fieldhouse [6]. Fieldhouse [7] showed the conjectured equality is valid for $d = 2m$ if it is valid for $d = 2m - 1$, and Grünbaum [11] used a theorem of Kruskal [21] to establish the validity for $d = 7$. Thus equality holds in (3) if any of the following conditions is satisfied:

$$(4) \qquad d \leq 8, \quad \text{or} \quad n \leq d + 3, \quad \text{or} \quad n \geq (d/2)^2 - 1.$$

For more detailed information on this and related questions see the papers cited above, the reports of Grünbaum [10] and Klee [19], and the forthcoming book by Grünbaum [11].

In addition to their obvious occurrence in linear programming, polyhedra arise in less obvious ways from other optimization methods, and results on the number of vertices are of interest for some of these. As an example we mention a procedure which has been applied in convex programming, best approximation and optimal controls (see Cheney & Goldstein [2] and some of their references). A subset X of R^d is said to be a <u>Haar set</u> provided X consists of at least $d + 1$ points and every d-pointed subset of X is a linear basis for R^d. In some of the problems discussed by Cheney & Goldstein [2] one is faced with a Haar set X, with a real-valued function φ defined on the set $\underset{\sim}{B}(X)$ of all positive bases for R^d contained in X, and with the task of minimizing φ over $\underset{\sim}{B}(X)$. The difficulty of this task depends in part on the cardinality $b(X)$ of $\underset{\sim}{B}(X)$, and thus it is natural to seek the maximum $M(d,h)$ of $b(X)$ as X ranges over all Haar sets of cardinality h in R^d. Let $C(X)$ denote the set of all convex relations on X --- that is, the set of all functions γ on X to $[0,1]$ such that $\Sigma_{x\epsilon X}\, \gamma_x = 1$ and $\Sigma_{x\epsilon X}\, \gamma_x\, x = 0$. Then $C(X)$ is a polytope of dimension $h - d - 1$ which has at most h facets, and by a theorem of Davis [5] there is a biunique correspondence between the vertices of $C(X)$ and the members of $\underset{\sim}{B}(X)$. Consequently

$$M(d,h) \leq \mu(d,h)$$

and (from (4))

$$(5) \qquad M(d,h) \le \binom{h-\left\lfloor \frac{h-d}{2} \right\rfloor}{d+1} + \binom{h-\left\lfloor \frac{h-d+1}{2} \right\rfloor}{d+1}$$

if $d \le 2$ or $h \le d + 9$ or $h \le d + 3 + 2(2d + 3)^{\frac{1}{2}}$

It is conjectured that equality always holds in (5), but this has been established only for $d \le 2$ and some other special case

The problem of determining the maximum length of the simple paths admitted by polyhedra of a given class appears to be equivalent to that of determining the maximum number of vertices. At any rate, the polytopes polar to cyclic polytopes have been shown to admit Hamiltonian circuits (Klee [18]), so if any of the conditions of (4) is satisfied then

$$\binom{n-\left\lfloor \frac{d+1}{2} \right\rfloor}{n-d} + \binom{n-\left\lfloor \frac{d+2}{2} \right\rfloor}{n-d} - 1$$

is the maximum of the lengths of simple paths admitted by polyhedra of class (d,n).

4. DIAMETERS

The known values for the functions Δ and Δ_b are tabulated below.

Δ

d \ n-d	1	2	3	4	5	
1	1	------	------	------	------	
2	*	2	3	4	5	. . . $\Delta(2,n) = n - 2$
3		*	3	4	5	. . . $\Delta(3,n) = n - 3$
4			*	5	?	
5				*	?	

Δ_b

d \ n-d	1	2	3	4	5	6	
1	1	------	------	------	------	------	
2	*	2	2	3	3	4	. . . $\Delta_b(2,n) = \lceil n/2 \rceil$
3		*	3	3	4	5	. . . $\Delta_b(3,n) = \lceil 2n/3 \rceil - 1$
4			*	4	5	?	
5				*	5	?	
6					*	?	

The formulas for $\Delta(2,n)$ and $\Delta_b(2,n)$ are obvious. The formula for $\Delta_b(3,n)$ and the fact that $\Delta_b(4,8) = 4$ were established by Klee [13], as was the formula for $\Delta(3,n)$ [17,18]. The asterisks in the tables indicate that each column is constant from the main diagonal downwards. This fact, the values of $\Delta(4,8)$, $\Delta_b(4,9)$ and $\Delta_b(4,10)$, and several other properties of the functions Δ and Δ_b, were established by Klee & Walkup [20]. It had been conjectured (p. 160 of Dantzig [4]) that $\Delta(d,n) = n - d$. While this is true for $d \leq 3$, note that $\Delta(4,8) = 5$. It was proved in [20] that

$$\Lambda(d,n) \geq n - d + \min\left(\left|\frac{d}{4}\right|, \left|\frac{n-d}{4}\right|\right)$$

so that the excess of $\Lambda(d,2d)$ over the conjectured value tends to infinity with d. The problem of determining $\Delta(d,n)$ for $d \geq 4$, or even of determining the asymptotic behavior of $\Delta(4,n)$ as $n \to \infty$, appears to be very difficult.

The assertions that $\Delta_b(d,n) \leq n - d$ and $\Delta_b(d,2d) = d$ were called in [20] the *bounded Hirsch conjecture and the bounded d-step conjecture* (cf. Danzig [4, p. 160 and p. 168]). As is seen from the second table, the bounded Hirsch conjecture has been established for $d \leq 3$ and the bounded d-step conjecture for $d \leq 5$. A related conjecture, due to Wolfe and Klee [18], asserts that any two vertices of a polytope can be joined by a so-called W_v path --- that is, by a path which does not revisit any facet from which it has earlier departed. This was proved in [17,18] for $d \leq 3$, and for 3-polyhedra as well as 3-polytopes. However, it fails for 4-polyhedra (though perhaps not for 4-polytopes) because $\Delta(4,8) = 5$ and on a d-polyhedron with n facets each W_v path is of length at most $n - d$. If φ is a linear function on a 3-polyhedron P, any vertex of P can be joined to a solution by a φ-path which is also a W_v path [17]. If the two vertices of a 3-polytope do not share a facet they can be joined by the three independent W_v paths (Barnette [1]). It was proved by Klee & Walkup [20] that the bounded Hirsch conjecture, the bounded d-step conjecture and the Wolfe-Klee conjecture are all equivalent, though not necessarily on a dimension-for-dimension basis.

5. HEIGHTS OF POLYTOPES

We turn now to some questions which are more directly related to linear programming, and specifically to the number of iterations which may be encountered for feasible regions of a given class in applying various rules for the formation of φ-paths leading to a solution. When $(x_0, x_1, \cdots, x_\ell)$ is a φ-path on a polyhedron P it is said to be a strict φ-path provided for $1 \le i \le \ell$, x_i is chosen so as to maximize φ on the vertices of P adjacent to x_{i-1}. The path is a steep φ-path provided for $1 \le i \le \ell$, x_i is chosen among the vertices adjacent to x_{i-1} so as to maximize the slope

$$\frac{\varphi(x_i) - \varphi(x_{i-1})}{\|x_i - x_{i-1}\|},$$

where $\|\ \|$ is the norm for the containing space. And the path is a simplex φ-path provided it is formed according to the standard simplex algorithm, requiring that x_i should be chosen among the vertices adjacent to x_i so as to maximize the so-called gradient of φ in the space of nonbasic variables. (See Dantzig [4, pp. 156-160] and Klee [15] for an explanation of this.) The third rule is the one most commonly used in practice because of its computational simplicity. It is geometrically more complicated than the others because the space in which the gradient is being maximized changes from one iteration to the next.

Let $L^t(d,n)$, $L^p(d,n)$, $L^x(d,n)$ and $L(d,n)$ denote the maximum length of strict φ-paths, steep φ-paths, simplex φ-paths and

φ-paths on polytopes of class (d,n). Here the maxima are taken over all polytopes P of the given class, all linear functions φ on P, and all φ-paths of the specified type. It was proved by Klee [15,16] that

$$(6) \qquad L^t(2,n) = n - 2, \ L^t(3,n) = \left\lfloor \frac{3n-1}{2} \right\rfloor - 4,$$

$$L^t(d,n) \geq 2(n-d) - 1 \quad \text{if} \quad d \geq 4,$$

$$(7) \qquad \left.\begin{array}{l} L^p(d,n) \\ L^x(d,n) \\ L(d,n) \end{array}\right\} \geq (n-d)(d-1) + 1, \quad \text{with equality if} \quad d \leq 3.$$

Comparison of the results for L^x and L suggest that at its worst the usual simplex algorithm behaves at least as badly as any of its variants, and comparison of the number $(n-d)(d-1)$ in (7) with the number $2(n-d)$ in (6) suggests the superior efficiency of linear programming algorithms which produce strict φ-paths. However, no firm conclusions can be drawn until much more evidence has been accumulated.

REFERENCES

1. D. Barnett, W_v paths in 3-polytopes, Journal of Combinatorial Theory, to appear.

2. E.W. Cheney and A.A. Goldstein, Newton's method for convex programming and Tchebycheff approximation, Numerische Math. 1(1959) 253-268.

3. G. B. Dantzig, Ten unsolved problems, hectographed notes, University of California, Berkeley, 1962.

4. ____________, Linear programming and extensions, Princeton Univ. Press, Princeton, N.J., 1963.

5. C. Davis, Theory of positive linear independence, Amer. J. Math. 76(1954) 733-746.

6. M. Fieldhouse, Linear programming, Ph.D. thesis, Cambridge Univ., 1961. (Reviewed in Operations Research 10 (1962) 740).

7. _________, Some properties of simplex polytopes, Dittoed notes, Harvard Univ., 1962.

8. D. Gale, Neighborly and cyclic polytopes, Proceedings of Symposia in Pure Math., vol. 7, Convexity, Amer. Math. Soc., Providence, R.I., 1963, pp. 225-232.

9. ______, On the number of faces of a convex polytope, Canad. J. Math. 16(1964) 12-17.

10. B. Grünbaum, On the number of faces of convex polytopes, Proceedings of a Colloquium on Convexity, Copenhagen, 1965, to appear.

11. __________, Convex polytopes, Wiley and Sons, New York, to appear.

12. W. W. Jacobs and E.D. Schell, The number of vertices of a convex polytope, (abstract), Amer. Math. Monthly 66(1959) 643.

13. V. Klee, Diameters of polyhedral graphs, Canad. J. Math. 16(1964) 602-614.

14. ______, The number of vertices of a convex polytope, Canad. J. Math. 16(1964) 701-720.

15. ________, Heights of convex polytopes, J. Math. Anal. Appl. 11(1965) 176-190.

16. ________, A class of linear programming problems requiring a large number of iterations, Numerische Math 7 (1965) 313-321.

17. ________, Paths on polyhedra. I, J. Soc. Ind. Appl. Math. 13(1965) 946-956.

18. ________, Paths on polyhedra. II. Pacific J. Math. 16(1966), to appear.

19. ________, Convex polytopes and linear programming. Proceedings of the IBM Scientific Computing Symposium on Combinatorial Problems, March 16-18, 1964, to appear.

20. V. Klee and D. Walkup, The d-step conjecture for polyhedra of dimension $d < 6$, to appear.

21. J.B. Kruskal, The number of simplices in a complex, Mathematical optimization techniques, Univ. of California Press, Berkeley, 1963, pp. 251-278.

22. H.W. Kuhn and R.E. Quandt, An experimental study of the simplex method, Proceedings of Symposia in Applied Math., vol. 15, Experimental Arithmetic, Amer. Math. Soc., Providence, R.I., 1963, pp. 107-124.

23. T.S. Motzkin, Comonotone curves and polyhedra, (Abstract 111), Bull. Amer. Math. Soc. 63(1957) 35.

24. R.E. Quandt and H.W. Kuhn, On some computer experiments in linear programming, Bulletin de l'Institut International de Statistique 1962, 363-372.

25. ____________________, On upper bounds for the number of iterations in solving linear programs, Operations Research 12(1964) 161-165.

26. T.L. Saaty, A conjecture concerning the smallest bound on the iterations in linear programming, Operations Research 11(1963) 151-153.

27. P. Wolfe and L. Cutler, Experiments in linear programming, Recent advances in mathematical programming, McGraw-Hill, New York, 1963, pp. 177-200.